arXiv:hep-th/9409188v1 29 Sep 1994

hep-th/9409188, HUTP-94/A034, IASSNS-HEP-94-69

ON ORBIFOLDS WITH DISCRETE TORSION

Cumrun Vafa

Lyman Laboratory of Physics, Harvard University
Cambridge, MA 02138 USA

and

School of Natural Sciences, Institute for Advanced Study
Olden Lane, Princeton, NJ 08540, USA

and

Edward Witten

School of Natural Sciences, Institute for Advanced Study
Olden Lane, Princeton, NJ 08540, USA

We consider the interpretation in classical geometry of conformal field theories constructed from orbifolds with discrete torsion. In examples we can analyze, these spacetimes contain "stringy regions" that from a classical point of view are singularities that are to be neither resolved nor blown up. Some of these models also give particularly simple and clear examples of mirror symmetry.

October, 1994

1. Introduction

Much of the fascination and mystery of string theory involves the relation between classical and stringy geometry. One facet of this involves singularities of space-time; what in classical geometry is a singularity may in string theory simply be a region in which stringy effects are large. An elementary example involves the classical solutions of string theory constructed as orbifolds [1], by which we mean in this context simply the quotient of a torus M by a finite group Γ. If Γ does not act freely, the classical space-time has singularities at fixed points of elements of Γ, but the conformal field theory of the orbifold is nonetheless perfectly regular.

Particularly interesting is the case in which M is a complex torus, say of complex dimension n, and Γ preserves the complex structure of M and acts trivially on the canonical line bundle. Then the conformal field theory of M/Γ has $(2,2)$ supersymmetry and integral $U(1)$ charges, just like conformal field theories associated with smooth Calabi-Yau manifolds. In fact, in simple cases one readily finds in the spectra of the orbifold theory in the twisted sectors marginal operators with the quantum numbers of elements of $H^1(M,T)$ or $H^1(M,T^*)$, suggesting that the orbifold can be blown up or resolved to get a smooth Calabi-Yau manifold. This was originally noticed by hand in some simple examples [1]; the results have been extended in various directions

[2,3,4]. When such resolution or blow-up is possible, the orbifold theory has the status of a soluble special case of a (generically nonsoluble) family of conformal field theories associated with the smooth Calabi-Yau manifold.

Orbifold conformal field theories, however, have a generalization by turning on what is known as discrete torsion [5], which involves introducing non-trivial phases to weight differently certain path-integral sectors. These non-trivial phases can be introduced when $H^2(\Gamma,U(1))\neq 0$. The issues mentioned above have not been addressed in the context of conformal field theory with discrete torsion; it is our intention to begin this analysis here. The result we will find (in the examples we will analyze) is as follows. Addition of discrete torsion markedly changes the geometrical interpretation of an orbifold theory; in many instances, discrete torsion gives a theory that is not continuously connected to a theory based on a smooth Calabi-Yau manifold. In those cases, by resolution or deformation one

can partially eliminate the singularities, but one remains with isolated singularities in the classical space-time. To a string theorist, these singularities are simply regions in which stringy effects are large. The discrete torsion is supported entirely in these isolated stringy regions.

For instance, from some viewpoints the simplest possible isolated singularity of a three-dimensional Calabi-Yau manifold is the conifold singularity

$$ {x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2 = 0. \tag{1.1} $$

In contrast to orbifold singularities, this type of singularity is a singularity even in conformal field theory – in the absence of discrete torsion. (For instance, Yukawa couplings have a pole at the conifold point [6].) We will find, though, that with $\mathbf{Z}_2$ discrete torsion, the conifold is *not* a singularity in conformal field theory.

A smooth Calabi-Yau manifold (without discrete torsion) can develop a conifold singularity through degeneration of either its complex structure or its Kahler structure. Both of these possibilities have arisen in conformal field theory, and in fact [7] they are

apparently mirror to each other. Conversely, the singularity of the conifold can be removed either by deformation of complex structure, that is, by deforming the equation to

$$ {x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2 = \epsilon, \tag{1.2} $$

with ϵ a complex parameter, or by resolving the singularity. (The relevant resolutions are the small resolutions that preserve the Calabi-Yau condition; they are described from the point of view of gauge theory in [8].) To summarize the situation, in the absence of discrete torsion there are two known conformal field theories associated with the conifold – one, call it the A model, in which the singularity arises by degeneration of Kahler structure (and theta angle),[1] and one, call it the B model, in which the singularity arises by degeneration of complex structure. Each model depends on one complex parameter, which measures the extent to which the Kahler structure (A model) or complex structure (B model) has been deformed. Such deformation is obligatory (in the A model case it is enough to deform the

[1] The Kahler structure in the sense of conformal field theory combines the conventional Kahler structure with the theta angles.

theta angle away from zero rather than deforming the classical Kahler structure) since the conifold is a singularity in conformal field theory.

In this paper, we will find a third conformal field theory of the conifold – call it the C model. In the C model, there is a $\mathbf{Z}_2$ discrete torsion sitting at the "singularity," which is in fact not a singularity in the conformal field theory sense, but just a region in which stringy effects are essential. The C model has no marginal operators (in particular the analogs of $H^{1,1}$ and $H^{2,1}$ are zero), so in the C model there is no way to modify or eliminate the "singularity." This is the opposite of the situation in the A and B models, in which deformation or blowup from the singularity are obligatory. The A model has $H^{1,1} = 1$, $H^{2,1} = 0$, and the B model has $H^{1,1} = 0$, $H^{2,1} = 1$, so transitions preserving $(2, 2)$ supersymmetry cannot occur among the A, B, and C models.

1.1. Mirror Symmetry

We have not yet mentioned another aspect of the present work which was in fact the starting point: the connection with mirror symmetry. In the course of analyzing simple examples of orbifolds with discrete torsion, we will find a very simple example of mirror symmetry, perhaps the only known type of example (apart from a complex torus) in which mirror symmetry can be understood and demonstrated completely.

This also raises the question of what mirror symmetry does to the C model. Apparently the "singularity" (or more properly the stringy region) of the C model has no invariant meaning: while conformal field theories approaching A type singularities are mirror to theories approaching B type singularities, those with C type singularities can be mirror to conformal field theories of perfectly smooth Calabi-Yau manifolds. To put this differently, we will get an example in which a maximally extended family of smooth Calabi-Yau manifolds is mirror to a family of Calabi-Yau manifolds that are all singular. This may be a much more general phenomenon for mirror symmetry.

2. A $\mathbf{Z}_2 \times \mathbf{Z}_2$ Orbifold

We need orbifolds with a finite group Γ such that $H^2(\Gamma, U(1))$ is non-zero, so that

discrete torsion is possible. We will take $\Gamma = \mathbf{Z}_k \times \mathbf{Z}_k$; in fact, in our examples, k will mainly be 2 or 3.

One has $H^2(\mathbf{Z}_k \times \mathbf{Z}_k, U(1)) \cong \mathbf{Z}_k$, so discrete torsion is possible. Let us describe precisely how it is implemented in conformal field theory, even though it may be familiar to many readers, as this will facilitate our discussion of constructing explicit orbifold models with discrete torsion. The conformal field theory with target space M is constructed in terms of maps of $\Sigma \to M$, with Σ a Riemann surface. For the conformal field theory of the orbifold M/Γ, one must consider maps of $\Sigma \to M/\Gamma$. A map to M/Γ can be regarded as a map to M which, in looping around a non-contractible path in Γ, is "twisted" by elements of Γ.[2] For instance, suppose Σ is of genus one; for instance, let Σ be the quotient of the $\sigma - \tau$ plane by $\sigma \to \sigma + m$, $\tau \to \tau + n$, with $m, n \in \mathbf{Z}$. A map of Σ to M/Γ involves twists by elements of $\Gamma = \mathbf{Z}_k \times \mathbf{Z}_k$ in both the σ and τ directions. If ζ denotes a generator of $\mathbf{Z}_k$ – we will identify ζ with the complex number $\exp(2\pi i/k)$ – then the σ and τ twists involve elements of Γ that we can write

$$\begin{aligned} T_\sigma &= (\zeta^a, \zeta^b) \\ T_\tau &= (\zeta^{a'}, \zeta^{b'}). \end{aligned} \tag{2.1}$$

Discrete torsion for Γ can now be described explicitly. Pick an integer $m = 0, 1, \ldots, k-1$. In the path integral of the orbifold, weight a sector with given twists T_σ, T_τ by an extra factor

$$\epsilon(T_\sigma, T_\tau) = \zeta^{m(ab'-ba')}. \tag{2.2}$$

Thus, depending on the choice of m, there are k distinct possible sets of weights in the path integral. The "usual" theory is $m = 0$, and the $k-1$ distinct non-zero choices for m give the theories with discrete torsion. The formula (2.2) has a unique generalization to genus g that is compatible with factorization in any channel. In fact this can be described once we choose a marking on a Riemann surface involving a canonical basis of 1-cycles (a_i, b_i). Relative to this marking the twisting can be described by thinking of a, b, a', b' above as corresponding to g-dimensional vectors. In that case the formula (2.2) for the discrete torsion is still valid where we think of the products as inner products of vectors.

[2] Because Γ is abelian, the twist involves a well determined element of Γ; otherwise one would get only a conjugacy class.

Hamiltonian Formulation

So far we have described the path integral realization of discrete torsion. It is also convenient to know how discrete torsion appears in the Hamiltonian formulation. Let us think of τ as "time" on the toroidal world-sheet Σ, and σ as "space." The configurations at $\tau = 0$ are classified by the twist T_σ. By quantizing the configurations of given T_σ, one constructs a Hilbert space $\mathcal{H}_{T_\sigma}$; this is called the Hilbert space in the sector twisted by T_σ. In the path integral, in addition to summing over T_σ, one also sums over T_τ. The summation over T_τ gives a projection onto the Γ-invariant part of $\mathcal{H}_{T_\sigma}$.

From this point of view, what is the meaning of the extra factor (2.2) in the path integral? In fact, the group Γ acts on $\mathcal{H}_{T_\sigma}$, but this action is not uniquely determined. There is a "standard" action on the Hilbert space that comes from the action of Γ on the space of configurations; let us call this standard representation $T_\tau \to \widehat{T}_\tau$. But a new representation can be defined by

$$\widehat{T}'_\tau = \widehat{T}_\tau \cdot \epsilon(T_\sigma, T_\tau). \tag{2.3}$$

Indeed, since ϵ has the property $\epsilon(x, yz) = \epsilon(x, y)\epsilon(x, z)$, including this factor – for any fixed T_σ – still leaves us with a representation of Γ. In the path integral with the factor (2.2), the sum over T_τ corresponds to a projection onto Γ-invariant states, using the Γ action in (2.3).

If ${\mathcal{H}_{T_\sigma}}^\Gamma$ is the Γ-invariant part of $\mathcal{H}_{T_\sigma}$, then overall the complete Hilbert space is

$$\mathcal{H} = \oplus_{T_\sigma} {\mathcal{H}_{T_\sigma}}^\Gamma. \tag{2.4}$$

2.1. An Example

Now we consider our first real example: a $\mathbf{Z}_2 \times \mathbf{Z}_2$ orbifold in complex dimension three.

For $i = 1 \dots 3$, let z_i be a complex variable, L_i a lattice in the z_i plane, and $E_i = \mathbf{C}/L_i$ the quotient of the z_i plane by L_i; of course, E_i is a Riemann surface of genus one. Set $T = E_1 \times E_2 \times E_3$. Let Γ be the group of symmetries of T consisting of transformations of the form

$$z_i \to (-1)^{\epsilon_i} z_i, \tag{2.5}$$

with

$$\prod_i (-1)^{\epsilon_i} = 1. \tag{2.6}$$

This condition ensures that the holomorphic three-form $\omega = dz_1 \wedge dz_2 \wedge dz_3$ is Γ-invariant. Γ is isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2$, and has three non-trivial elements, each of which changes the sign of precisely two of the z_i.

We wish to consider the Calabi-Yau orbifold T/Γ. First we consider some simple facts about the classical geometry. The operation $z_i \to -z_i$ on the torus E_i has four fixed points. Using this, it is easy to work out the fixed point sets of the non-trivial elements of Γ. They are all similar, so we may as well consider the element α that acts as -1 on z_1 and z_2 and as $+1$ on z_3. Since there are four invariant values of z_1, four invariant values of z_2, and the action on z_3 is trivial, the fixed point set of α consists of $4 \times 4 = 16$ copies of E_3. Since the two other non-trivial elements of Γ act similarly, the set W on which Γ does not act freely is a union of $3 \times 16 = 48$ tori. We will call them fixed tori although each such torus is fixed only by a $\mathbf{Z}_2$ subgroup of Γ.

So far we have classified the points in T that are invariant under one $\mathbf{Z}_2$ subgroup of $\Gamma = \mathbf{Z}_2 \times \mathbf{Z}_2$. The other singular orbits correspond to points that are left fixed by all of Γ, in other words points invariant under $z_i \to -z_i$ for $i = 1, 2, 3$. The number of such points is $4 \times 4 \times 4 = 64$. Each of the 64 fixed points lies at the intersection of three fixed tori; for instance, the fixed point $z_1 = z_2 = z_3 = 0$ is the intersection of the torus $z_1 = z_2 = 0$, the torus $z_1 = z_3 = 0$, and the torus $z_2 = z_3 = 0$.

Cohomology

To understand some essential properties of this orbifold in string theory, let us compute the spectrum of ground states in the Ramond (R) sector[3]; this gives the analog of what for a smooth Calabi-Yau manifold would be the cohomology.

First we consider the untwisted sector. The cohomology of any one of the E_i is

[3] Here and in the following, we take periodic boundary conditions for left- and right-moving fermions; thus what we call the Ramond sector is sometimes called the Ramond-Ramond or RR sector.

described by the Hodge diamond

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{2.7}$$

The numbers displayed here are the dimensions of $H^{p,q}(E_i)$, with p being the vertical axis and q the horizontal axis. The transformation $z_i \to -z_i$ acts as $+1$ on $H^{0,0}$ and $H^{1,1}$ and as -1 on $H^{1,0}$ and $H^{0,1}$. This follows from the fact that $H^{0,0}$, $H^{1,1}$, $H^{1,0}$ and $H^{0,1}$ are generated respectively by the differential forms 1, $dz_i \wedge d\overline{z}_i$, dz_i , and $d\overline{z}_i$.

The Ramond ground states coming from the untwisted sector are simply the Γ-invariant part of $H^*(E_1 \times E_2 \times E_3) = H^*(E_1) \times H^*(E_2) \times H^*(E_3)$. With $H^*(E_i)$ as described in the previous paragraph, the Γ-invariant part of $H^*(E_1 \times E_2 \times E_3)$ is readily determined and can be summarized by the Hodge diamond

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 3 & 3 & 0 \\ 0 & 3 & 3 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}. \tag{2.8}$$

For instance, $H^{1,1}$ is three-dimensional, generated by $dz_i \wedge d\overline{z}_i$, for $i = 1,2,3$, $H^{2,1}$ is three-dimensional, generated by such forms as $dz_1 \wedge dz_2 \wedge d\overline{z}_3$, and $H^{3,0}$ is one-dimensional, generated by the holomorphic three-form ω.

To complete the picture, we must determine the spectrum of R ground states from the twisted sectors. There are three twisted sectors since the abelian group Γ has three non-trivial elements. As we noted above, each non-trivial element acts as -1 on precisely two of the three z_i, so we may as well consider a group element α that acts as -1 on z_1 and z_2 and as $+1$ on z_3. We must find the Γ-invariant R ground states in the Hilbert space $\mathcal{H}_\alpha$ of strings twisted by α.

Classically, to have zero energy a string should be a constant configuration, that is independent of the spatial string coordinate σ. In a sector twisted by α, the constant must be a fixed point of α. The element α has four fixed points in acting on E_1, four in acting on E_2, and of course acts trivially on E_3. So the fixed point set of α consists of sixteen copies of E_3. Quantization of the space of constant strings gives the space of R ground states in $\mathcal{H}_\alpha$; it is a sum of sixteen copies of the cohomology of E_3. Before writing the corresponding contribution to the Hodge diamond, that is the dimensions of the $H^{p,q}$, we

must remember that p and q arise in the physics as certain $U(1)$ charges and that in the twisted sectors the zero point values of p and q receive certain shifts which are needed to ensure Poincaré duality (or CPT invariance) of the orbifold theory.[4] The shifts mean that the sixteen copies of $H^{p,q}(E_3)$ contribute to $H^{p+1,q+1}$ of the orbifold theory, so that the R ground states in $\mathcal{H}_\alpha$ can be described by the Hodge diamond

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 16 & 16 & 0 \\ 0 & 16 & 16 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.9}$$

It remains to extract the Γ-invariant subspace of these twisted R ground states. In the absence of discrete torsion, we take the natural Γ action on $H^{p,q}(E_3)$. Since Γ acts on E_3 through $z_3 \to -z_3$, in the natural action of Γ, the forms 1 and $dz_3 \wedge d\overline{z}_3$ are invariant and dz_3 and $d\overline{z}_3$ are not. So the $H^{1,1}$ and $H^{2,2}$ contributions in (2.9) survive as contributions to the ground state spectrum of the orbifold. Since there are three twisted sectors, obtained by permutations of the z_i from the one we have analyzed, the total contributions of the twisted sectors to the dimensions of $H^{1,1}$ and $H^{2,2}$ are $3\times 16 = 48$. Adding these figures to the untwisted Hodge diamond of equation (2.8), the Hodge diamond of the orbifold theory without discrete torsion is

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 3 & 51 & 0 \\ 0 & 51 & 3 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}. \tag{2.10}$$

It remains to consider the case with discrete torsion. Using the explicit description of discrete torsion for $\mathbf{Z}_n \times \mathbf{Z}_n$ in (2.2), it can be seen that for the Γ element α that we have considered, $\epsilon(\alpha,\beta) = -1$ precisely if $\beta \neq 1, \alpha$. With the particular $\mathbf{Z}_2 \times \mathbf{Z}_2$ action that we have taken on $T = E_1 \times E_2 \times E_3$, β acts on z_3 as $z_3 \to -z_3$ precisely if $\beta \neq 1, \alpha$. Putting these facts together, the effect of the discrete torsion in the sector twisted by α is simply to include an extra minus sign in the transformation of the states under $z_3 \to -z_3$. With the new transformation law, dz_3 and $d\overline{z}_3$ are even and $1, dz_3 \wedge d\overline{z}_3$ are odd. So with

[4] If the complex coordinates z_i are twisted by $z_i \to e^{i\theta_i} z_i$ with $0 \leq \theta_i < 2\pi$, then the zero point shift in p and q is by $\sum_i \theta_i/2\pi$ [1]. This has been studied in detail in [9]. All assertions in this paper about zero point shifts follow from this formula. In the present example, the θ_i are $\pi, \pi, 0$ so the shift is by 1.

discrete torsion, the part of (2.9) that contributes to the cohomology of the orbifold is $p = 2, q = 1$ and vice-versa. Including the three twisted sectors, the cohomology of the orbifold is therefore described by the Hodge diamond

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 51 & 3 & 0 \\ 0 & 3 & 51 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}. \tag{2.11}$$

Notice that (2.10) and (2.11) are mirrors of each other, that is related by $p \leftrightarrow 3-p$, $q \leftrightarrow q$; this is the operation that reverses the sign of the left-moving $U(1)$ charge without affecting the right-moving one. This hints that the theories with and without discrete torsion are mirror; this is true, as we will explain in section 3.

With or without discrete torsion, the twisted sectors added 48 states to $H^{1,1}$ or $H^{2,1}$. Each such state arose as the contribution of one of the 48 fixed tori in T. Each of these tori becomes a $\mathbf{Z}_2$ orbifold singularity in T/Γ. These singularities are of complex codimension two and look locally like the singularity

$$y^2 = uv. \tag{2.12}$$

This singularity can either be deformed away, by adding a parameter to the equation to give, say,

$$y^2 = uv + \epsilon, \tag{2.13}$$

or it can be blown up. In terms of the three-dimensional orbifold theory, the deformation involves a mode in $H^{2,1}$ supported along the fixed torus, and the blowup involves a mode in $H^{1,1}$ that is supported there.

From our computation of the spectrum, it is clear what is happening. Without discrete torsion, the twisted sector modes are in $H^{1,1}$ and the fixed torus singularity is blown up; with discrete torsion, the twisted sector modes are in $H^{2,1}$ and the fixed torus singularity is deformed.

Blowing up or deformation of the fixed tori removes the singularities of the orbifold T/Γ in complex codimension two. But what happens to the 64 fixed points – is one left with singularities in complex codimension three? In the case of the blow-up – that is,

without discrete torsion – the answer is that blowing up the codimension two singularities automatically also eliminates the singularities in codimension three. This is related to the fact that for abelian groups, the conformal field theory of orbifolds agrees with classical geometry [4],[5] as a result of which in this paper we will have little to say about the theory without discrete torsion.

With discrete torsion, one encounters deformation rather than blowup, and the answer is quite different, as we will see.

2.2. Deformation Of The Orbifold

We wish to compare the orbifold conformal field theory with discrete torsion to the classical geometry obtained by deforming the singularities of the orbifold $T = (E_1 \times E_2 \times E_3)/\Gamma$.

An alternative description of the genus one surfaces E_i is convenient. A Riemann surface E of genus one can be described as a double cover of $\mathbf{CP}^1$ branched over four points; it can be described, therefore, by an equation $y^2 = F(u, v)$ where u, v are homogeneous coordinates of $\mathbf{CP}^1$, y is homogeneous of degree 2, and F is a homogeneous quartic polynomial. $\mathbf{Z}_2$ acts on E by $y \to -y$, with u, v invariant. The fixed points of $\mathbf{Z}_2$ are therefore the four homogeneous solutions of $F(u, v) = 0$.

So $T = E_1 \times E_2 \times E_3$ can be described by variables u_i, v_i, y_i, $i = 1 \dots 3$, with equations

$$ {y_i}^2 = F_i(u_i, v_i). \tag{2.14} $$

To give an algebraic description of the quotient T/Γ, we simply identify the Γ-invariant sub-ring of the ring of polynomial functions in the u_i, v_i, and y_i. Noting that Γ acts by pairwise sign changes of the y_i, the invariants are the u_i, the v_i, and $y = y_1 y_2 y_3$ (y is homogeneous of degree two with respect to scalings of any pair u_i, v_i), subject to the one equation

$$ y^2 = \prod_{i=1}^{3} F_i(u_i, v_i). \tag{2.15} $$

[5] The analysis of [4] does not quite apply to the present $\mathbf{Z}_2 \times \mathbf{Z}_2$ example, because of the codimension two singularities, but the extension to the $\mathbf{Z}_2 \times \mathbf{Z}_2$ example has been described by the author of [4].

Now – leaving physics aside – it is clear how to deform the complex structure of T/Γ to get a smooth Calabi-Yau manifold. One simply deforms the function $\prod_i F_i(u_i, v_i)$ to a generic polynomial $F(u_1, v_1; u_2, v_2; u_3, v_3)$ which is homogeneous of degree four in each pair of variables u_i, v_i. The equation

$$y^2 = F(u_i, v_i) \tag{2.16}$$

describes a double cover of $\mathbf{CP}^1 \times \mathbf{CP}^1 \times \mathbf{CP}^1$ which – for generic F – is a smooth Calabi-Yau manifold.

Let us count the number of parameters in this family of Calabi-Yau manifolds. The space of quartic polynomials in two variables u, v is of dimension 5. The space of polynomials F of degree four in each pair is therefore of dimension $5 \times 5 \times 5 = 125$. From this number we should remove $3 \times 3 = 9$ corresponding to the action of $SL(2, \mathbf{C})$ on each pair u_i, v_i, and 1 for overall scaling of F (which can be absorbed in scaling of y). So the total number of polynomial deformations is $125 - 9 - 1 = 115$. Though more complicated phenomena occur for other Calabi-Yau manifolds (see [10], Chapter A, for an introduction), in the present example, it can be readily shown that the polynomial deformations faithfully represent the possible deformations of the complex structure.

2.3. The Meaning Of The Discrepancy

From these calculations, we get a discrepancy between the orbifold with discrete torsion and the classical geometry. The orbifold with discrete torsion is part of a family of conformal field theories that depends on 51 "complex structure" moduli, while the smooth Calabi-Yau manifold that is obtained by deformation of complex structure of T/Γ has 115 complex structure moduli. The difference is $115 - 51 = 64$.[6]

But 64 is a number that we have already seen: it is the number of $\mathbf{Z}_2 \times \mathbf{Z}_2$ fixed points. As we have already discussed, these are the points that may remain as codimension three singularities after the codimension two singularities are deformed away.

[6] On the other hand, it can be seen, for instance by computing the Euler characteristic, that the smooth Calabi-Yau manifold given by a generic equation (2.15) has $b^{1,1} = 3$, in agreement with the conformal field theory of the orbifold with discrete torsion.

Let us first explain the interpretation that we wish to propose for the discrepancy. A generic equation (2.16) depends on 115 parameters and describes a smooth Calabi-Yau manifold. Suppose that one does not wish to get a smooth Calabi-Yau manifold but one with certain singularities. Then restrictions must be placed on the parameters, so the most general Calabi-Yau manifold in this family with specified singularities will depend on fewer than 115 parameters. The precise number depends on the type of singularities one prescribes.

Consider a singularity of a three-dimensional complex manifold described by a general equation $F(x_1, x_2, x_3, x_4) = 0$, with an isolated singularity at the origin, where $F = \partial_i F = 0$. Suppose one considers an arbitrary small perturbation to a nearby equation

$$F(x_1, x_2, x_3, x_4) = \epsilon(x_1, x_2, x_3, x_4). \tag{2.17}$$

Terms in ϵ that can be expressed as linear combinations (with holomorphic coefficients) of the partial derivatives $\partial_i F$ can be transformed away by shifting the x_i. (For instance, $\epsilon = \epsilon_0 \partial_1 F$ is removed to first order by $x_1 \to x_1 + \epsilon_0$.) So the space of relevant operators is the space of polynomials in the x_i modulo the ideal generated by the derivatives $\partial_i F$. (This is familiar to string theorists in the Landau-Ginzburg theory of singularities [11,12].) The identity operator is always relevant. There are additional relevant operators unless the x_i are all in the ideal, which happens precisely if F is equivalent locally to the conifold

$$F = {x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2. \tag{2.18}$$

Thus, the conifold is the unique isolated singularity with precisely one relevant operator. If one wishes to have precisely two relevant operators, then the relevant operators must be 1 and a linear function [7] of the x_i, say x_1; the ideal must then contain x_2, x_3, x_4, and ${x_1}^2$ (or there would be more than two relevant operators). One must then have up to a choice of coordinates

$$F = {x_1}^3 + {x_2}^2 + {x_3}^2 + {x_4}^2. \tag{2.19}$$

[7] By a linear function we mean really a function with only a first order zero; it becomes a linear function if coordinates are chosen correctly. If there are two relevant operators, one must be linear since if the x_i are all in the ideal, so are all higher order polynomials and the identity is the only relevant operator.

In a suitably generic family of Calabi-Yau manifolds, a singularity with k relevant operators will appear in complex codimension k, since to obtain that singularity one must adjust k relevant parameters. For instance, a conifold singularity will (generically) arise in complex codimension one. Thus, if one wishes to deform (2.15) to an equation describing an (otherwise generic) Calabi-Yau manifold with a conifold point, the number of parameters will be 114 instead of 115. If one wants n disjoint conifold singularities, the number of parameters is $115 - n$. For n singularities each with k relevant operators, the number of parameters will be $115 - kn$.

In our problem, we start with 48 fixed tori and 48 twisted sector marginal operators that represent deformations of those codimension two singularities. This leaves unclear whether there will be codimension three singularities at the 64 fixed points. If so, the number of complex structure parameters of the Calabi-Yau will be $115 - 64n$, where n is the number of relevant operators of the singularity. (With appropriate complex structures on the E_i, there are symmetries permuting the fixed points, so n is the same for each.) The actual number of complex structure deformations of the conformal field theory with discrete torsion is $51 = 115 - 64$, so this will fit if $n = 1$. But $n = 1$ corresponds precisely to the conifold. Thus, we get a candidate for the geometrical interpretation of the conformal field theory of the orbifold with discrete torsion: it corresponds to a Calabi-Yau with 64 conifold singularities.

This is a conifold that cannot be eliminated by any marginal operator of the conformal field theory, since the 64 requisite operators are missing. It is as if discrete torsion supported at the conifold singularity prevents it from being deformed. We will discuss the physical interpretation more fully at the end of this section.

Local Behavior

To support our explanation of the apparent discrepancy between conformal field theory and geometry, we will examine some of the above-mentioned ingredients more carefully. First we look at the behavior near one of the fixed points.

In terms of the description of the E_i by equations ${y_i}^2 = F_i(u_i, v_i)$, we can take the fixed point to be at $u_1 = u_2 = u_3 = 0$. By scaling, one can set $v_1 = v_2 = v_3 = 1$. The fixed

points are zeroes of the F_i, so we can assume that near $u_i = 0$, $F_i \sim u_i$. Thus equation (2.15) takes the form

$$y^2 = u_1 u_2 u_3 \tag{2.20}$$

near the fixed point. Note that in (2.20), there are codimension two singularities on three curves C_i – one with $u_1 = u_2 = 0$, another with $u_1 = u_3 = 0$, and another with $u_2 = u_3 = 0$. The fixed point at $u_1 = u_2 = u_3 = 0$ is the intersection of these three lines of codimension two singularities. Of course, this is expected since more globally the fixed points are intersections of three fixed tori.

Now the first point is that it is possible to add a perturbation to (2.20) that eliminates the codimension two singularities but leaves a conifold singularity at the origin. This could be simply

$$y^2 = u_1 u_2 u_3 + \epsilon({u_1}^2 + {u_2}^2 + {u_3}^2). \tag{2.21}$$

A general first order perturbation

$$y^2 = u_1 u_2 u_3 + \epsilon(u_1, u_2, u_3) \tag{2.22}$$

removes the codimension two singularities if ϵ is generically non-zero on the C_i. It in addition removes the singularity at the origin – where the C_i intersect – if $\epsilon(0,0,0) \neq 0$.

Global Story

Now let us examine precisely which 64 modes are missing. We start with the unperturbed equation

$$y^2 = F_1(u_1, v_1) F_2(u_2, v_2) F_3(u_3, v_3) \tag{2.23}$$

with the F_i being homogeneous quartic polynomials.

Each F_i takes values in a five dimensional space V_i of homogeneous quartics. F_i itself generates a one dimensional subspace $V_{i,0}$ of V_i; let W_i be a four dimensional complement to $V_{i,0}$. We will take the symbol δF_i to refer to a variation of F_i that is constrained to lie in W_i.

We perturb (2.23) to an equation of the form

$$y^2 = F_1 F_2 F_3 + \epsilon \delta F. \tag{2.24}$$

The possible δF's can be classified as follows.

There is one variation in which the change in $F_1F_2F_3$ is a multiple of itself. This is irrelevant since it can be absorbed in rescaling y. Our notation does not even permit us to write this mode conveniently since we require δF_i to take values in the complementary space W_i.

There are twelve deformations of the form $\delta F = \delta F_1\, F_2F_3 + F_1\delta F_2\, F_3 + F_1F_2\delta F_3$ in which only one of the F_i is deformed. However $3 \times 3 = 9$ of them can be removed by $SL(2, \mathbf{C})$ transformations on (u_i, v_i). Altogether, then, there are $12 - 9 = 3$ non-trivial modes of this kind. These deformations give equations that in first order in ϵ are equivalent to $y^2 = (F_1+\epsilon\delta F_1)(F_2+\epsilon\delta F_2)(F_3+\epsilon\delta F_3)$. This is of the general form of (2.15) with different F_i, so it describes an orbifold T/Γ with a different complex structure on T. Comparing to conformal field theory, these are the three modes from the untwisted sector that preserve the orbifold structure.

There are 48 modes in which two of the F_i are varied. These are modes of the form

$$\delta F = \delta F_1\delta F_2\, F_3 + F_1\delta F_2\delta F_3 + \delta F_1\, F_2\delta F_3. \tag{2.25}$$

These modes have the property that they vanish on the 64 fixed points (which are characterized by $F_1 = F_2 = F_3 = 0$) but they do not generically vanish on the fixed tori (on which only two of the F_i vanish). Therefore, the modes of this form remove the codimension two singularities but in first order leave codimension three singularities at the fixed points.

Finally, there are 64 modes in which all three F_i are varied,

$$\delta F = \delta F_1\delta F_2\delta F_3. \tag{2.26}$$

These modes have no particular zeroes and would remove all the singularities. We claim, however, that these are the modes that are missing in the conformal field theory.

This is strongly suggested by the structure of the computation of the twisted sector modes in the conformal field theory. Each twisted sector mode comes from a sector twisted by a group element similar to the element α that acts non-trivially on the two elliptic curves E_1 and E_2 and trivially on E_3. The modes in the α-twisted sector should deform the singularity of the α-fixed tori, that is the singularities with $F_1 = F_2 = 0$. They should

not deform the singularities from fixed tori of other group elements. The modes that do this must vanish when $F_1 = F_3 = 0$ or $F_2 = F_3 = 0$ (so as not to disturb the fixed tori of other group elements) but not when $F_1 = F_2 = 0$. These modes must therefore be of the form $\delta F_1 \delta F_2 F_3$. Similarly, the other twisted sectors give the other modes in (2.25). But nothing in the conformal field theory gives the 64 modes in (2.26).

Support Of The Torsion

Before discussing the support of the torsion, we need to recall a generality about discrete torsion. In discrete torsion, one starts with an element γ of $H^2(\Gamma, U(1))$, and (as long as one keeps away from singularities) this is then mapped to an element $\widehat{\gamma}$ of $H^2(M_0, U(1))$, with M_0 the smooth part of the orbifold T/Γ; physically, the world-sheet theory then has a B-field in the cohomology class of $\widehat{\gamma}$. If $\widehat{\gamma}$ is in the connected component of $H^2(M_0, U(1))$, this B field is described as a world-sheet theta angle; if not, it is called discrete torsion in space-time. Both possibilities can arise [13].

Returning to our problem, let p be a fixed point in the orbifold T/Γ. The underlying discrete torsion is non-trivial in an arbitrarily small neighborhood of p since it appears in the weight of the path integral for string world-sheets that sit arbitrarily close to (or even at) the singularity.

After the complex deformation is made, in the presence of the underlying discrete torsion, a neighborhood of p looks like the complex singularity

$$x_1{}^2 + x_2{}^2 + x_3{}^2 + x_4{}^2 = 0 \tag{2.27}$$

with p being the point $x_i = 0$. If one deletes the conifold point p from this space, one gets a smooth manifold W with the homology of $\mathbf{S}^2 \times \mathbf{S}^3$.[8] In particular, $H^2(W, \mathbf{Z}) \cong \mathbf{Z}$ and $H^2(W, U(1)) \cong U(1)$. As this is connected, the underlying discrete torsion of the orbifold could not produce discrete torsion of W, but it might produce a theta angle. To test this possibility, we need to know whether the discrete torsion produces a phase for a

[8] If one treats the variables in (2.27) as homogeneous variables, the equation describes a smooth quadric in $\mathbf{CP}^3$, isomorphic to $\mathbf{CP}^1 \times \mathbf{CP}^1$. In projectivizing the variables, one divides by a $\mathbf{C}^*$ action, so the conifold with the origin deleted is a $\mathbf{C}^*$ bundle over $\mathbf{CP}^1 \times \mathbf{CP}^1$. The cohomology can be computed using this fibration.

string world-sheet that wraps around a generator $\mathbf{S}$ of $H_2(W, \mathbf{Z}) \cong \mathbf{Z}$. One can choose $\mathbf{S}$ to be a two-sphere obtained in resolving the fixed tori of the orbifold; in that operation a point with $z_1 \neq 0$, $z_2 = z_3 = 0$ is replaced by a two-sphere $\mathbf{S}$. But for a world-sheet with z_1 almost constant and non-zero, the discrete torsion (which only contributes in sectors in which all three z_i are twisted non-trivially) does not give any phase. Therefore, the underlying discrete torsion of the orbifold does not contribute anything if restricted to W.

To summarize all our statements, the underlying discrete torsion produces an effect which is non-zero in an arbitrarily small neighborhood of p, but zero if p is deleted; it is roughly as if the discrete torsion has delta function support at p. One can roughly model the situation by supposing that, with the proper definition, the conifold M has a torsion element in $H^2(M, U(1))$ which would disappear if the conifold singularity is deformed or resolved; then discrete torsion supported at the conifold point would explain the inability to deform the conifold. However, we do not know the proper definition of $H^2(M, U(1))$ to justify this interpretation.

In any event, what is going on at the conifold singularity can not necessarily be properly interpreted as a discrete torsion with delta function support. It is not at all clear that the conifold theory that we have found, which does make sense, differs just by discrete phases from the more traditional (singular) A and B models of the conifold. Our argument started with an orbifold with discrete torsion, which differed from the ordinary orbifold only by such phases, but by the time we deform to the conifold, there is no way to compare the model to another model that is "identical except for phases."

As another interpretation, perhaps in an infinitesimal neighborhood of the singularity an H-field is turned on (recall $H = dB$ where B is the two form). This may be natural from the following viewpoint. If there were no fixed points, inclusion of discrete torsion is equivalent to turning on a B-field. Just as the orbifolds have curvature singularities concentrated at the fixed points, it may be that orbifolds with discrete torsion have the torsion field H concentrated at the fixed points. Moreover this may also explain why the singularity cannot be deformed. It can be shown that with an H field turned on in a smooth way the $N = 2$ superconformal symmetry is broken [14]. It may be the case that with delta function support the $N = 2$ superconformal symmetry may be restored. In fact

this may be the reason that in the context of $N = 2$ superconformal theory with torsion there is no marginal deformation that gets rid of the singularity.

A similar phenomenon occurs in the context of bosonic string orbifolds. In the case of bosonic strings, conformal invariance for smooth manifolds favors that the manifold should be flat. This however is in contrast with explicit toroidal orbifold conformal theories which are flat everywhere except for delta function curvature singularities. One would thus expect that in the case of bosonic string there are no marginal operators that get rid of singularities of orbifolds; this is indeed generally the case. Thus bosonic orbifolds provide examples of isolated singularities that cannot be deformed–very much as discrete torsion on the conifold behaves in the superstring case.

To summarize this, all we really know is that the underlying discrete torsion produces an effect – an H field or something else – that is supported at the singularity and whose presence makes the singularity inescapable. It would be very interesting to find a more explicit description of this quantum field theory and learn what is going on.

3. Mirror Symmetry and $\mathbf{Z}_2 \times \mathbf{Z}_2$ Orbifold with Torsion

In this section we will show that the $\mathbf{Z}_2 \times \mathbf{Z}_2$ orbifold described in the previous section provides a simple realization of mirror symmetry: the orbifold with discrete torsion is mirror to the same orbifold without discrete torsion. We have already found a hint of this in observing that the Hodge diamonds of these models are mirror of one another. Here we will show that they indeed are identical superconformal theories. This implies, in particular, that the complex structure deformation of the $\mathbf{Z}_2 \times \mathbf{Z}_2$ orbifold with discrete torsion described in the last section is mirror to the Kahler deformation of the same orbifold theory without torsion. This also implies that by studying the periods of the deformation discussed there one should be able to deduce the quantum cohomology ring for the blown-up orbifold. This computation should be interesting to do as it would provide a further check on the geometrical interpretation of the orbifold with discrete torsion advanced in the last section.

In order to show that the two orbifold theories with and without discrete torsion are equivalent one should show an identification between the operators of the two theories

under which the correlation functions and partition functions coincide. It is generally considered sufficient to show that the spectra are identical in the two cases and that in addition they have the same three point correlators on the sphere, or alternatively, that the partition functions at all genera are identical. In the case at hand, all these things are easy to prove.

3.1. Mirror Symmetry and Phase of Path-Integral

Mirror symmetry is an isomorphism between two $(2,2)$ conformal field theories under which the sign of the left-moving $U(1)$ charge, but not the right-moving one, is reversed. Roughly, this means that the complex structure is reversed for left-movers but not for right-movers. As is well known, in the case of a three-fold such a symmetry flips the sign of $\mathrm{Tr}(-1)^{\mathrm{F}}$ for the supersymmetric ground states.

A basic result in this area is that the mirror of a complex torus is another complex torus. Let us recall how this comes about. Begin by considering a free scalar field Y, compactified on a circle of radius R. It is well known that this theory is equivalent to a similar theory with radius $1/R$; under this transformation there is a non-trivial identification of operators:

$$\begin{aligned} \partial Y &\to \partial Y \\ \overline{\partial} Y &\to -\overline{\partial} Y. \end{aligned} \tag{3.1}$$

Now add a second free periodic boson X, of radius R'. We suppose that the metric is just $ds^2 = dX^2 + dY^2$ so that a local complex coordinate is just $Z = X + iY$. Consider the transformation $R \to 1/R$ for the Y variable, without doing anything to X. (The B field must vanish to make this a symmetry.) Obviously, as the operators ∂X and $\overline{\partial} X$ are invariant, one has

$$\begin{aligned} \partial Z &\to \partial Z \\ \overline{\partial} Z &\to \overline{\partial}\, \overline{Z}. \end{aligned} \tag{3.2}$$

Thus, the complex structure is reversed for left-movers but not for right-movers; this is a mirror symmetry. Note in particular that the volume (RR') and the shape (R'/R) get exchanged under $R \to 1/R$. Thus a two dimensional rectangular torus with zero B field and radii R', R is mirror to a similar model with radii $R', 1/R$. By following the possible deformations on the two sides, one learns that any two-torus is mirror to another two-torus.

There is no problem in generalizing this to higher dimensions. One simply considers several periodic variables X_i, several Y_i, and $Z_i = X_i + iY_i$. A transformation $R \to 1/R$ on each of the Y_i brings about

$$\begin{aligned} \partial Y_i &\to \partial Y_i \\ \overline{\partial} Y_i &\to -\overline{\partial} Y_i, \end{aligned} \tag{3.3}$$

and so

$$\begin{aligned} \partial Z_i &\to \partial Z_i \\ \overline{\partial} Z_i &\to \overline{\partial}\ \overline{Z}_i. \end{aligned} \tag{3.4}$$

So this transformation is a mirror symmetry.

Now, let us consider precisely how this mirror symmetry acts on the fermions. Each X_i and each Y_i is related by world-sheet supersymmetry to a right-moving fermion and to a left-moving fermion. Since the right-moving parts of X_i and Y_i are invariant under the mirror symmetry, world-sheet supersymmetry implies that the right-moving fermions are also invariant. The same is true for the left-moving partners ψ_i of X_i. However, as the left-moving part of Y_i has its sign reversed under $R \to 1/R$, the left-moving partners η_i of Y_i change sign in this operation. Thus, if we are in complex dimension three, the mirror symmetry reverses the sign of precisely three left-moving fermions.

In fact, this lets us check that this operation is a mirror symmetry. The operator $(-1)^{F_L}$ in the Ramond sector (the left-moving part of $(-1)^F$) contains a zero mode part that is the product of the zero modes of ψ_i and η_i. As there are an odd number of η's, the reversal in sign of the η_i gives the expected sign change of $(-1)^{F_L}$ under mirror symmetry.

Path Integral Formulation

Let us discuss what this looks like from a path integral point of view. On world-sheet fermions, we may impose antiperiodic (A) or periodic (P) boundary conditions in circling around a string. This leads to four boundary conditions or spin structures for genus 1 depending on the boundary conditions in the σ and τ directions; we may call these (P, A), (A, A), (A, P), and (P, P). Of these, (P, P) is modular invariant, and the other three are permuted by modular transformations. In genus g there are 2^{2g} spin structures.

A spin structure is said to be even or odd depending on whether the number of negative (or positive) chirality fermion zero modes is even or odd. For instance, in genus one, (P, P)

is odd (there is one zero mode, the constant), and the others are even (there are no zero modes).

The importance for us of counting the number of zero modes is that this determines the behavior of the world-sheet path integral measure under mirror symmetry. We have seen that mirror symmetry acts by $\eta_i \rightarrow -\eta_i$ (with other fermions invariant), so the measure is even or odd depending on whether the number of components of η_i is even or odd. The non-zero modes are naturally paired, so the measure is even or odd depending on whether the number of zero modes of the η_i is even or odd. If therefore α denotes the spin structure of left-moving fermions and σ_α is 0 or 1 for α an even or odd spin structure, then for target space a torus the genus g measure $\mu_{g,\alpha}$ with spin structure α transforms under mirror symmetry as

$$\mu_{g,\alpha} \rightarrow (-1)^{\sigma_\alpha} \mu_{g,\alpha}. \tag{3.5}$$

The Orbifold

Now we consider a situation with three $Z_i = X_i + iY_i$, and we introduce the group $\Gamma = \mathbf{Z}_2 \times \mathbf{Z}_2$, acting, as in section 2, by pairwise sign changes of the Z_i.

Since the group Γ commutes with the transformation (3.3) of the Y_i under $R \rightarrow 1/R$, the $R \rightarrow 1/R$ operation can be done for the orbifold T/Γ, not just for the original torus T. However, just as for the original torus, the $R \rightarrow 1/R$ transformation induces a sign change $\eta \rightarrow -\eta$, and we must determine what this sign change does to the path integral measure. We will do this explicitly in genus one before discussing the generalization.

To do the genus one path integral of the orbifold, we have to consider the path integral with toroidal target and various twisted boundary conditions in the σ and τ directions. Let g and h be the elements of Γ that act on (Z_1, Z_2, Z_3) as $(1, -1, -1)$ and $(-1, 1, -1)$, respectively. A general twist would involve a pair of group elements $(x, y) = (g^r h^s, g^t h^u)$, and as stated in equation (2.2), the discrete torsion for this pair corresponds to a sign factor

$$\epsilon(g^r h^s, g^t h^u) = (-1)^{ru-st}. \tag{3.6}$$

We want to show that for fermions with a given spin structure α and a given set of $\mathbf{Z}_2 \times \mathbf{Z}_2$

twists (x, y), the path integral measure transforms under $R \to 1/R$ by

$$\mu_\alpha(g, h) \to (-1)^{\sigma_\alpha} \epsilon(x, y) \mu_\alpha(g, h). \tag{3.7}$$

If true, this asserts that the orbifold theory without (or with) discrete torsion transforms under $R \to 1/R$ into the mirror theory with (or without) discrete torsion. The factor $(-1)^{\sigma_\alpha}$ makes the transformation a mirror transformation, and the second factor is the discrete torsion.

Up to modular transformations and permutations of the Z_i, it is enough to check (3.7) for $(x, y) = (1, 1)$, $(g, 1)$, and (g, h). We already know that the result is true for $(1, 1)$. Let us consider (g, h).

Suppose that $\sigma_\alpha = 1$, that is, suppose that the fermions are in the odd or periodic spin structure (P, P). The (g, h) twist reverses the boundary conditions for the η_i (supersymmetric partners of the Y_i) in the σ or τ directions and effectively shifts the η_i into the even spin structures (P, A), (A, P), and (A, A). So the measure with $\sigma_\alpha = 1$ and twists (g, h) is even; this agrees with (3.7).

Now keep the twists (g, h) but take $\sigma_\alpha = 0$. It suffices to consider the spin structure (A, A) as the other two even spin structures are related to this by a modular transformation (which preserves (g, h) up to a permutation of the Z_i).

In this case, the effect of the (g, h) twist is to shift η_3 into the odd spin structure while leaving the others in even spin structures. So, as one fermion is in an odd spin structure, the path integral measure is odd, as predicted by the above formula, for even spin structure and twists $(x, y) = (g, h)$.

To complete the verification of (3.7), it remains to consider the case of twists $(g, 1)$. In this case, $\epsilon = 1$ and we must show that the measure transforms as $(-1)^{\sigma_\alpha}$. For instance, for the (P, P) spin structure, $\sigma_\alpha = 1$, the $(g, 1)$ twist leaves η_1 with effective (P, P) couplings and shifts the others to (A, P); hence there is one fermion zero mode and the measure is odd under $\eta \to -\eta$. For spin structure (A, P), $\sigma_\alpha = 0$, the twist by $(g, 1)$ puts an even number of fermions (η_2 and η_3) in the odd spin structure, so the measure is even; for (P, A) and (A, A), the $(g, 1)$ twist leaves all fermions in even spin structures, so the measure is again even. This completes the verification of (3.7).

So we have verified that for the orbifold, $R \to 1/R$ on the Y_i has the following characteristics: (i) it is a mirror operation, because of the factor of $(-1)^{\sigma_\alpha}$; (ii) it adds (or removes) discrete torsion, because of the factor of ϵ. Having understood how the measure transforms under $R \to 1/R$, there is almost nothing to add to discuss correlation functions. If operators $\mathcal{O}_i$ transform into operators $\mathcal{O}_i'$ under $R \to 1/R$, then the correlation functions of the $\mathcal{O}_i$ without discrete torsion are equal to the correlation functions of the $\mathcal{O}_i'$ with discrete torsion. This is true in genus one from the above analysis of the measure, and it is true more trivially in genus zero where there is only an even spin structure. To verify that the assertion is true in higher genus, we will presently analyze the path integral measure in arbitrary genus.

Generalization To Genus g

To determine how the measure transforms in genus g, we will need some further facts about spin structures in higher genus (see [15] for a review). Consider a genus g Riemann surface Σ^g. Choose a canonical basis of the 1-cycles labeled by (a_i, b_i) where $i = 1, ..., g$. Such a "marking" determines a canonical spin structure. Relative to this marking, any other spin structure is determined by twisting the fermions by signs ± 1 around the various a_i and b_j cycles.

Spin structures can therefore be classified by $\mathbf{Z}_2$-valued quantities θ_i, ϕ_j, $i, j = 1 \ldots g$, as follows: in the spin structure $\alpha = (\theta_i, \phi_j)$, fermions are twisted by an extra minus sign (relative to the canonical spin structure determined by the marking)

in going around any cycle a_i or b_j such that θ_i or ϕ_j is 1. It is sometimes convenient to think of θ_i, ϕ_i as g-dimensional vectors which we label by Θ, Φ. A classic formula says that the parity of the spin structure α is

$$\sigma_\alpha = \sigma(\Theta, \Phi) = \Theta \cdot \Phi \qquad \bmod 2 \tag{3.8}$$

That is, the number of fermion zero modes in the spin structure α is equal mod two to $\Theta \cdot \Phi = \sum_i \theta_i \phi_i$.

Now we consider the path integral of the orbifold theory for a given spin structure $\alpha = (\Theta, \Phi)$, with twists around the a_i cycle given by $g^{A_i} h^{C_i}$ and twists around the b_i

cycle given by $g^{B_i}h^{D_i}$. The A_i, B_i, C_i, D_i are integers defined mod 2; we combine them in vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$. The discrete torsion in this case is given according to equation (2.2) by

$$\epsilon = (-1)^{\mathbf{A}\cdot\mathbf{D}-\mathbf{B}\cdot\mathbf{C}} \tag{3.9}$$

As above, we want to show that under $\eta \to -\eta$, the path integral measure transforms by

$$\mu \to \mu \cdot (-1)^{\sigma(\Theta,\Phi)}\epsilon. \tag{3.10}$$

Using the definition of g and h, the effect of the twist is to shift the effective spin structures for the fermions in the Z_1, Z_2, Z_3 directions to

$$Z_1 : \qquad (\Theta + \mathbf{C}, \Phi + \mathbf{D})$$

$$Z_2 : \qquad (\Theta + \mathbf{A}, \Phi + \mathbf{B})$$

$$Z_3 : \qquad (\Theta + \mathbf{A} + \mathbf{C}, \Phi + \mathbf{B} + \mathbf{D})$$

The path integral measure therefore transforms by

$$\mu \to \mu \cdot (-1)^{\sigma(\Theta+\mathbf{C},\Phi+\mathbf{D})+\sigma(\Theta+\mathbf{A},\Phi+\mathbf{B})+\sigma(\Theta+\mathbf{A}+\mathbf{C},\Phi+\mathbf{B}+\mathbf{D})}. \tag{3.11}$$

Using (3.8) and (3.9), this coincides with the desired transformation law (3.10).[9] This completes the proof in the path integral formulation of how mirror symmetry acts for the $\mathbf{Z}_2 \times \mathbf{Z}_2$ orbifold.

3.2. Generalizations

The above can be generalized in two ways: one can use a different lattice with $\mathbf{Z}_2 \times \mathbf{Z}_2$ symmetry, or one can replace $\mathbf{Z}_2 \times \mathbf{Z}_2$ with a different group.

---

[9] In fact a classic theorem (stated as Theorem 2 in [16]) asserts that

$$\sigma(\Theta + \mathbf{A} + \mathbf{C}, \Phi + \mathbf{B} + \mathbf{D}) = \sigma(\Theta, \Phi) + \sigma(\Theta + \mathbf{A}, \Phi + \mathbf{B}) + \sigma(\Theta + \mathbf{C}, \Phi + \mathbf{D}) + (\mathbf{A}\cdot\mathbf{D} - \mathbf{B}\cdot\mathbf{C})$$

modulo 2. This formula, which directly relates the above expressions, is more or less equivalent to (3.8), but without need to choose a marking.

For the first generalization, one can replace the hypercubic lattice that we have used for simplicity with another lattice that preserves the essential properties of the construction. Those properties are the $\Gamma = \mathbf{Z}_2 \times \mathbf{Z}_2$ action preserving the complex structure and holomorphic three-form, and an additional operation ($Y_i \to -Y_i$ in the above) that reverses the complex structure and commutes with $\mathbf{Z}_2 \times \mathbf{Z}_2$. The hypercubic lattice is not the only one with these properties. Some others enter in [17]. In some of these cases, one obtains orbifolds with Hodge numbers different from the one discussed above.

The other generalization involves replacing $\mathbf{Z}_2 \times \mathbf{Z}_2$ with another group Γ. We will briefly describe how this can be done in substantial (but not complete) generality[10]. Let L be any three dimensional lattice in $\mathbf{R}^3$ and $\Gamma \subset SO(3)$ a group of symmetries of L. Let S be the three dimensional torus $S = \mathbf{R}^3/L$, and let S' be a second copy of S. Consider the six dimensional torus $T = S \times S'$. Let X_i and Y_i be (local) linear coordinates on S and S', respectively, and pick the complex structure on T such that $Z_i = X_i + iY_i$ are local complex coordinates. T is a Calabi-Yau manifold with holomorphic three-form $dZ_1 \wedge dZ_2 \wedge dZ_3$. Consider the diagonal action of Γ on T; that is, Γ acts on both factors S and S'. Then Γ preserves the complex structure and holomorphic three-form of T, so we can consider the Calabi-Yau orbifold T/Γ.

Γ also commutes with the operation $Y_i \to -Y_i$ so as in the above discussion of $\mathbf{Z}_2 \times \mathbf{Z}_2$, an $R \to 1/R$ transformation on S' can be used to show that the mirror of T/Γ is again T/Γ, with of course an inverted radius of S' and possibly a different discrete torsion. As for $\mathbf{Z}_2 \times \mathbf{Z}_2$, the discrete torsion that is generated by the mirror transformation can be computed by seeing how the path integral measure transforms under $\eta_i \to -\eta_i$. We will simply state the results without proof.

A set of Γ twists determines a flat $SO(3)$ bundle E, via the embedding $\Gamma \subset SO(3)$. For a given negative chirality spin bundle α, the η_i are a section of $\alpha \otimes E$. The path integral measure transforms as $(-1)^n$, where n is the number modulo 2 of zero modes of the Dirac operator coupled to $\alpha \otimes E$. That number is a topological invariant of E regarded

[10] All these considerations generalize to n-folds where we replace $SO(3)$ with $SO(n)$. In particular the discrete torsion contains an element that comes from $H^2(SO(n), \mathbf{Z}_2)$, which is related to how the twist elements lift from $SO(n)$ to $Spin(n)$.

as an $SO(3)$ bundle – in computing it one can ignore the finer structure that E has as a Γ-bundle. So the index can only depend on E through its one topological invariant, which is the second Stieffel-Whitney class $w_2(E) \subset \mathbf{Z}_2$. In fact, one can show (for instance, by deforming E to a sum of line bundles) that $n = \sigma_\alpha + w_2(E)$. The transformation of the measure under $\eta_i \to -\eta_i$ is thus

$$\mu_{\alpha,E} \to (-1)^{\sigma_\alpha}(-1)^{w_2(E)}\mu_{\alpha,E}. \tag{3.12}$$

As before, the factor of $(-1)^{\sigma_\alpha}$ means that this is a mirror symmetry, and the factor $(-1)^{w_2(E)}$ means that the mirror symmetry shifts the discrete torsion. In fact, $(-1)^{w_2(E)}$ is the E dependent phase factor associated with a particular element $x \in H^2(\Gamma, U(1))$. Under mirror symmetry, the discrete torsion is multiplied by x; the T/Γ theory with discrete torsion y is mirror to the same theory with discrete torsion xy.

One way to describe the torsion element $x \in H^2(\Gamma, U(1))$ more explicitly is as follows. We will describe an element $\widehat{x} \in H^2(SO(3), U(1))$ which, for any $\Gamma \subset SO(3)$, restricts to the required x. $H^2(SO(3), U(1))$ classifies extensions of $SO(3)$ by $U(1)$. One such extension is the group $U(2)$; that is, the center of $U(2)$ is isomorphic to $U(1)$ and the quotient $U(2)/U(1)$ is isomorphic to $SO(3)$:

$$1 \to U(1) \to U(2) \to SO(3) \to 1. \tag{3.13}$$

The element of $H^2(SO(3), U(1))$ associated with this extension is the desired $\widehat{x}$.

4. A $\mathbf{Z}_3 \times \mathbf{Z}_3$ Example

In this section, we will describe a $\mathbf{Z}_3 \times \mathbf{Z}_3$ orbifold that is somewhat similar to the $\mathbf{Z}_2 \times \mathbf{Z}_2$ example. To begin with, we need a genus one curve E with a $\mathbf{Z}_3$ symmetry that has non-trivial fixed points. This curve can be regarded as the complex z plane divided by a hexagonal lattice (its complex structure is uniquely determined since the hexagonal lattice is the only lattice in the plane with $\mathbf{Z}_3$ symmetry). The $\mathbf{Z}_3$ symmetry is generated by $z \to \zeta z$, with $\zeta = \exp(2\pi i/3)$.

Alternatively, the same curve can be described algebraically by the equation in homogeneous variables

$$y^3 = u^3 + v^3. \tag{4.1}$$

The $\mathbf{Z}_3$ symmetry is then generated by $y \to \zeta y$ (with u, v invariant). The $\mathbf{Z}_3$ action on E has three fixed points, the points with $y = 0$, $u^3 + v^3 = 0$.

Now we introduce three identical curves E_i, $i = 1 \dots 3$; E_i is the quotient of the z_i plane by a hexagonal lattice or alternatively is given by equations

$$y_i{}^3 = u_i{}^3 + v_i{}^3 \tag{4.2}$$

in homogeneous variables y_i, u_i, v_i. On $T = E_1 \times E_2 \times E_3$, there is a natural action of $\Gamma_0 = \mathbf{Z}_3 \times \mathbf{Z}_3 \times \mathbf{Z}_3$. The subgroup of Γ_0 that preserves the holomorphic three-form of T (which is $\omega = dz_1 \wedge dz_2 \wedge dz_3$) is the group of transformations $z_i \to \zeta^{a_i} z_i$ with $\zeta^{a_1 + a_2 + a_3} = 1$. We call this group Γ; it is isomorphic to $\mathbf{Z}_3 \times \mathbf{Z}_3$. We wish to study the orbifold T/Γ.

The possible discrete torsion in this theory can be described very explicitly. If T_σ and T_τ are two elements of Γ, say $T_\sigma = (\zeta^{a_1}, \zeta^{a_2}, \zeta^{a_3})$ and $T_\tau = (\zeta^{b_1}, \zeta^{b_2}, \zeta^{b_3})$, then the torsion factor in (2.2) is

$$\epsilon(T_\sigma, T_\tau) = \zeta^{m \sum_{i,j,k=1}^{3} \epsilon_{ijk} a_j b_k} \tag{4.3}$$

where ϵ_{ijk} is the antisymmetric tensor with $\epsilon_{123} = +1$.

The classical geometry can be studied similarly to the $\mathbf{Z}_2 \times \mathbf{Z}_2$ example. The group element $\alpha = (\zeta, \zeta^{-1}, 1)$ has a fixed point set consisting of nine tori. Allowing also for fixed tori of other group elements, there are $3 \times 9 = 27$ fixed tori in all. In addition, there are $3 \times 3 \times 3 = 27$ fixed points of the whole group; as in the $\mathbf{Z}_2 \times \mathbf{Z}_2$ example, these are intersections of three fixed tori.

A difference from the $\mathbf{Z}_2 \times \mathbf{Z}_2$ example is that in addition to the identity element and group elements that have fixed tori, $\mathbf{Z}_3 \times \mathbf{Z}_3$ also contains the elements $\beta = (\zeta, \zeta, \zeta)$ and β^2 that act with isolated fixed points (27 of them) rather than fixed tori.

4.1. Spectrum Of The Orbifold

Now let us determine the Ramond ground states of the orbifold. First we work in the absence of discrete torsion.

Each of the E_i has the familiar Hodge diamond

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{4.4}$$

In the $\mathbf{Z}_3$ action on the cohomology, H^0 and H^2 are invariant but $H^{1,0}$ (which is generated by dz_i) and $H^{0,1}$ (which is generated by $d\overline{z}_i$) transform with eigenvalue ζ or ζ^{-1}. The Γ-invariant part of the cohomology of $E_1 \times E_2 \times E_3$ can be represented by the Hodge diamond

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 3 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}. \tag{4.5}$$

This is the contribution of the untwisted sector to the cohomology of the orbifold.

Next we consider the sector twisted by α. The low-lying states are obtained by quantizing the fixed point set, which as we noted above consists of nine fixed tori. The spectrum of R ground states in the twisted Hilbert space $\mathcal{H}_\alpha$ consists therefore of nine copies of the cohomology of a torus; with the shift in the zero point values of the $U(1)$ charges, $H^{p,q}$ of the torus contributes to $H^{p+1,q+1}$ of the orbifold. The contribution of the nine tori can hence be represented by the Hodge diamond

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 9 & 9 & 0 \\ 0 & 9 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.6}$$

Now we must project onto the Γ-invariant part of the cohomology; in fact, the Γ-invariant states are those that contribute to $H^{1,1}$ and $H^{2,2}$ (coming from $H^{0,0}$ and $H^{1,1}$ of the torus). As there are altogether six group elements similar to α, the contribution to the Hodge diamond from this source is

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 54 & 0 \\ 0 & 54 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.7}$$

We also have the sectors twisted by β or β^2. The fixed point set consists in each case of 27 isolated points. The cohomology of a point consists of the one-dimensional space $H^{0,0}$ with trivial Γ action. But the shift in the zero point of the $U(1)$ charges shifts the

contribution of the β twisted sector to $H^{1,1}$ and that of the β^2 twisted sector to $H^{2,2}$. So the contribution to the Hodge diamond is

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 27 & 0 \\ 0 & 27 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.8}$$

Adding it all up, the Hodge diamond of the orbifold without discrete torsion is

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 84 & 0 \\ 0 & 84 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}. \tag{4.9}$$

In particular, there are 84 Kahler deformations and no complex structure deformations.

Inclusion Of Discrete Torsion

Now we repeat the analysis in the presence of discrete torsion. We may as well pick $m = 1$ in (4.3), since the $m = 2$ case differs by a permutation of the E_i.

The contribution to the space of R ground states from the untwisted sector is unaffected by discrete torsion since $\epsilon(T_\sigma, T_\tau) = 1$ if $T_\sigma = 1$.

Now let us consider the α-twisted sector. Though $\epsilon(\alpha, \alpha) = 1$, we have $\epsilon(\alpha, \widetilde{\alpha}) = \zeta^{-1}$ where $\widetilde{\alpha} = (1, \zeta, \zeta^{-1})$. Hence in the presence of discrete torsion, projecting onto the Γ-invariants means projecting onto states that in the natural action of Γ transform under $\widetilde{\alpha}$ as ζ. As $\widetilde{\alpha}$ acts on E_3 by $z_3 \to \zeta^{-1} z_3$, the only state in the cohomology of E_3 that transforms as ζ in the natural action of $\widetilde{\alpha}$ on the cohomology of E_3 is $d\overline{z}_3$, which generates $H^{0,1}$. Therefore (upon allowing for the shift in the zero point) the nine fixed tori of α, which are copies of E_3, contribute to $H^{1,2}$ of the orbifold. In all, of the six $\mathbf{Z}_3 \times \mathbf{Z}_3$ elements obtained from α by permutations of the E_i, three contribute to $H^{1,2}$ of the orbifold and three to $H^{2,1}$. The total contribution from these sectors is therefore

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 27 & 0 & 0 \\ 0 & 0 & 27 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.10}$$

Similarly, with discrete torsion, to get the contribution of the β-twisted sector we must project onto the part of the cohomology of the fixed point set of β that transforms

as ζ under α. But the fixed point set of β is a set of 27 points, which are all left fixed by α so that α acts trivially on their cohomology; hence, with discrete torsion, the β-twisted sector does not contribute any R ground states. The same goes for β^2.

The overall Hodge diamond of the theory with discrete torsion is therefore

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 27 & 3 & 0 \\ 0 & 3 & 27 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}. \tag{4.11}$$

This spectrum is not mirror to (4.9), which should come as no surprise since the construction described in section 3 does not apply.

4.2. Comparison To Classical Geometry

Since the conformal field theory without discrete torsion has Kahler deformations and no complex structure deformations, it should be compared (as in [4]) to the blow-up of the orbifold. On the other hand, with discrete torsion there are complex structure modes, and one wonders to what extent the conformal field theory with discrete torsion can be compared to the conformal field theory of a smooth Calabi-Yau manifold obtained by deforming T/Γ.

Let us find a family of smooth Calabi-Yau manifolds to which the orbifold T/Γ can be deformed. To this aim, we want an algebraic description of the orbifold. Beginning with the algebraic description (4.2) of the torus T, we simply project onto the Γ-invariant polynomials. They are u_i, v_i, and $y = y_1 y_2 y_3$ (y is of degree one under scaling of any pair u_i, v_i), obeying the single equation

$$y^3 = \prod_{i=1}^{3} ({u_i}^3 + {v_i}^3). \tag{4.12}$$

This can be deformed to

$$y^3 = F(u_i, v_i) + yG(u_i, v_i), \tag{4.13}$$

with F a function that is cubic in each pair of variables u_i, v_i and G quadratic in each pair. (This preserves the homogeneity of the equation. Note that a term $y^2 H(u_i, v_i)$ need

not be included as it could be eliminated by $y \to y + H/3$.) For generic F, G, we get a smooth Calabi-Yau manifold.

Let us count the parameters in (4.13). The space of quartic polynomials in u_i, v_i is of dimension 4. The space of cubic polynomials is of dimension 3. So the space of F's and G's is of dimension $4^3 + 3^3 = 91$. After removing one for an overall scaling and $3 \times 3 = 9$ to allow for the $SL(2, \mathbf{C})$ action on u_i, v_i, we are left with 81 parameters in the equations. These are the right parameters since in the present example the polynomial deformations can be shown to faithfully represent the possible deformations of the complex structure.

4.3. Origin Of The Discrepancy

So once again we have a discrepancy: the number of complex structure deformations in the conformal field theory of the orbifold (with discrete torsion) is 27, but in the classical geometry there are 81. The number of missing modes is 54, which equals 2×27, where 27 is the number of $\mathbf{Z}_3 \times \mathbf{Z}_3$ fixed points.[11]

So, as in the $\mathbf{Z}_2 \times \mathbf{Z}_2$ example, one can suspect that the singularities at the fixed points are not completely eliminated by the complex structure deformations, and that one instead gets 27 singularities each of which has 2 relevant operators. As we explained earlier, the only isolated singularity with precisely 2 relevant operators is

$$y^3 = {u_1}^2 + {u_2}^2 + {u_3}^2, \tag{4.14}$$

and if one wishes to have 27 singularities with this structure, one would have to impose $2 \times 27 = 54$ conditions on the parameters in (4.13). The proposal is then that the orbifold with discrete torsion (and generic complex structure deformation) corresponds to a specialization of (4.13) with 54 conditions imposed to ensure 27 singularities of this type.

As in the $\mathbf{Z}_2 \times \mathbf{Z}_2$ case, evidence for this interpretation can be found by considering in more detail the possible perturbations of (4.13). The space V_i of cubic polynomials in u_i, v_i is four-dimensional. Pick in this space a three-dimensional complement W_i to the

[11] There is no such discrepancy for the Kahler deformations. One can show – for instance by computing the Euler characteristic – that the smooth Calabi-Yau manifold given by a generic equation (4.13) has $b^{1,1} = 3$, in agreement with the conformal field theory with discrete torsion.

one-dimensional subspace generated by ${u_i}^3 + {v_i}^3$. Let $F_i = {u_i}^3 + {v_i}^3$, and let δF_i be an element of W_i.

Then – apart from modes that can be eliminated by scaling and $SL(2, \mathbf{C})$ – the modes in (4.13) can be written in detail as follows. There are $3 \times 3 \times 3 = 27$ modes in which to the unperturbed equation (4.12) one adds

$$\delta F_1 \delta F_2 F_3 + F_1 \delta F_2 \delta F_3 + \delta F_1 F_2 \delta F_3. \tag{4.15}$$

There are 27 more modes of the form

$$\delta F_1 \delta F_2 \delta F_3 \tag{4.16}$$

and 27 of the form

$$yG(u_i, v_i). \tag{4.17}$$

The 27 complex structure modes of the conformal field theory with discrete torsion each arose as a contribution of a particular fixed torus from a particular twisted sector. The mode associated with a given fixed torus must deform the singularity of that torus but not the others. The modes that do this are the ones in (4.15). For instance, any torus fixed by $\alpha = (\zeta, \zeta^{-1}, 1)$ lies at $F_1 = F_2 = 0$, so the associated modes should not vanish if $F_1 = F_2 = 0$, but should vanish if $F_1 = F_3 = 0$ or $F_2 = F_3 = 0$ (to avoid deforming other fixed tori). The modes in the α-twisted sector are $\delta F_1 \delta F_2 F_3$. Similarly, the other twisted sectors give the other modes in (4.15). These modes all vanish at the fixed points $F_1 = F_2 = F_3$ and do not have the flexibility to eliminate the singularities at the fixed points. The modes in (4.16) and (4.17) would do this, but do not arise in the conformal field theory.

The 27 surviving singularities are of the form (4.14) as this is the only isolated singularity with two relevant operators. As a check, let us note that in (4.14) there is a $\mathbf{Z}_3$ symmetry $y \to \zeta y$ (with the u_i fixed); the relevant operators 1 and y transform as 1 and ζ. Similarly, the orbifold theory (4.12) has a $\mathbf{Z}_3$ symmetry $y \to \zeta y$; the 54 missing perturbations include 27 modes (4.16) that are invariant and 27 modes (4.17) transforming as ζ. This is in agreement with what one would expect for 27 singularities of the structure claimed.

The Support Of The Torsion

For the same reasons as in the $\mathbf{Z}_2 \times \mathbf{Z}_2$ case, the discrete torsion is non-zero in an arbitrarily small neighborhood of the singularity in (4.14). On the other hand, the complement W of the singularity has the cohomology of $\mathbf{S}^5$ [18]. In particular, $H^2(W, U(1))$ vanishes and with it the discrete torsion. The effects of the underlying discrete torsion are thus supported at the origin.

In fact, from [18], one can make a stronger statement: the singular space (4.14) is topologically equivalent to $\mathbf{R}^6$, and thus we have found a classical solution of string theory that can be interpreted as a kind of stringy "lump" in $\mathbf{R}^6$, whose complement is isomorphic topologically to the complement of an ordinary ball in $\mathbf{R}^6$. However, the (singular) Calabi-Yau metric on the space (4.14) is presumably not asymptotic at infinity to the flat metric on $\mathbf{R}^6$.

We would like to thank A. Faraggi and K.S. Narain for discussions on $\mathbf{Z}_2 \times \mathbf{Z}_2$ orbifolds and P. Aspinwall, T. Hubsch, S. Katz and D. Morrison for useful advice about Calabi-Yau manifolds.

The research of C.V. was supported in part by the Packard fellowship and NSF grants PHY-92-18167 and PHY-89-57162. The work of E.W. was supported in part by NSF Grant PHY92-45317.

References

[1] L. Dixon, J. Harvey, C. Vafa, and E. Witten, "Strings On Orbifolds, I, II," Nucl.Phys. **B261** (1985) 678, **B274** (1986) 285.

[2] D. Markushevich, M. Olshanetsky, and A. Perelomov, "Description Of A Class Of Superstring Compactifications Related To Semisimple Lie Algebras," Commun. Math. Phys. **111** (1987) 247.

[3] S.S. Roan, "Minimal Resolutions of Gorenstein Orbifolds in Dimension Three," Acad. Sinica preprint R940606-1.

[4] P. Aspinwall, "$(2,2)$ Superconformal Field Theories Near Orbifold Points," Commun. Math. Phys. **128** (1990) 593, "Resolution Of Orbifold Singularities In String Theory," IASSNS-HEP-94/9.

[5] C. Vafa, "Modular Invariance And Discrete Torsion On Orbifolds," Nucl. Phys. **B273** (1986) 592.

[6] P. Candelas, X. della Ossa, P. S. Green, and L. Parkes, "A Pair Of Calabi-Yau Manifolds As An Exactly Soluble Superconformal Field Theory," Nucl. Phys. **B359** (1991) 21.

[7] D. Morrison, "Through The Looking Glass," Duke preprint, to appear.

[8] E. Witten, "Phases of $N = 2$ Models In Two Dimensions," Nucl. Phys.**B403** (1993) 159.

[9] E. Zaslow, "Topological Orbifold Couplings And Quantum Cohomology Rings," Comm. Math. Phys. **156** (1993) 301.

[10] T. Hubsch, *Calabi-Yau Manifolds: A Bestiary For Physicists* (World Scientific, 1992).

[11] C. Vafa and N. Warner, "Catastrophes And The Classification Of Superconformal Theories," Phys. Lett. **B218** (1989) 51.

[12] E. Martinec, "Algebraic Geometry And Effective Lagrangians," Phys. Lett. **B217** (1989) 431.

[13] P. Aspinwall and D. Morrison, "Chiral Rings Do Not Suffice: $N = (2,2)$ Theories With Nonzero Fundamental Group," DUK-TH-94-70, IASNSN-HEP-94/37.

[14] R. Rohm and E. Witten, "The Antisymmetric Tensor Field In String Theory," Ann.Phys. **170** (1986) 454.

[15] L. Alvarez-Gaume, G. Moore and C. Vafa, "Theta Functions, Modular Invariance, and Strings," Comm. Math. Phys. **106** (1986) 40.

[16] M. F. Atiyah, "Riemann Surfaces And Spin Structures," Ann. Sci. Ecole Normale Sup. **4** (1971) 47.

[17] A. Faraggi, "$\mathbf{Z}_2 \times \mathbf{Z}_2$ Orbifold Compactification As The Origin Of Realistic Free Fermionic Models," Phys. Lett. **B326** (1994) 62.

[18] J. Milnor, *Singular Points Of Complex Hypersurfaces,* (Princeton University Press, 1968), p. 72.